BAULKHAM HILLS HIGH SCHOOL

YEAR 11

HSC ASSESSMENT TASK

December 2010

MATHEMATICS
EXTENSION 1

Time allowed - 70 minutes

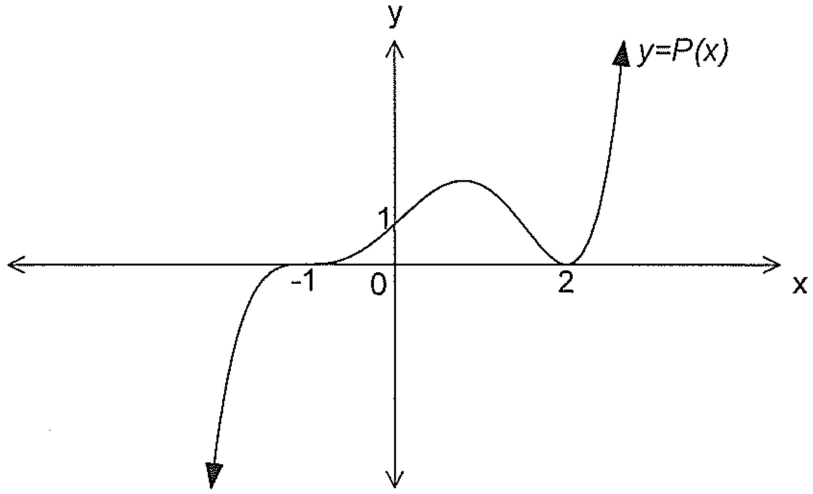
DIRECTIONS TO CANDIDATES

- Attempt ALL questions. You do NOT need to start each question on a new page.
- Allocated marks are indicated for each question.
- All necessary working should be shown.
- Approved calculators and Mathematical templates may be used
- Write your teacher's name and your name on the cover sheet provided.
- At the end of the exam, staple your answers in order behind the cover sheet provided. Staple the question sheets to the back.

Question 1

a) Given $P(x) = x^4 - x^2 + 1 \; and \; Q(x) = x^2 + 1$, divide $P(x)$ by $Q(x)$ Hence express $P(x)$ in the form $Q(x).A(x) + R(x)$, where $A(x) and \; R(x)$ are polynomials **2**

b) Find a possible equation for the curve shown **3**



Question 2

The polynomial $P(x) = 2x^3 - 5x^2 + 3x + 1$ has zeroes $\propto$, β and γ. Find the value of:

(i) $\alpha + \beta + \gamma$ **1**
(ii) $\alpha^{-1} + \beta^{-1} + \gamma^{-1}$ **2**
(iii) $\alpha^2 + \beta^2 + \gamma^2$ **2**

Question 3

a) Find the Cartesian equation of the curve with the parametric equations:

(i) $x = 2t, y = 3t^2$ **1**
(ii) $x = \cos\theta, y = sin\theta - 1$ **1**

b) $x^2 - 4$ is a factor of $x^4 + x^3 + px^2 + qx + 48$. Find the value of p and q. **3**

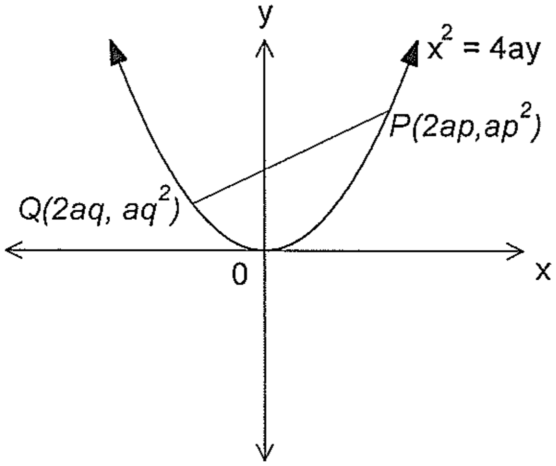
Question 4

Using the principle of Mathematical Induction, prove that

$$\sum_{r=1}^{n} 5^r = \frac{5}{4}(5^n - 1), \text{ for all positive integers n.}$$ **4**

Question 5

$P(2ap, ap^2)$ and $Q(2aq, aq^2)$ are two points on the parabola $x^2 = 4ay$.



a) Find the equation of the chord PQ **2**

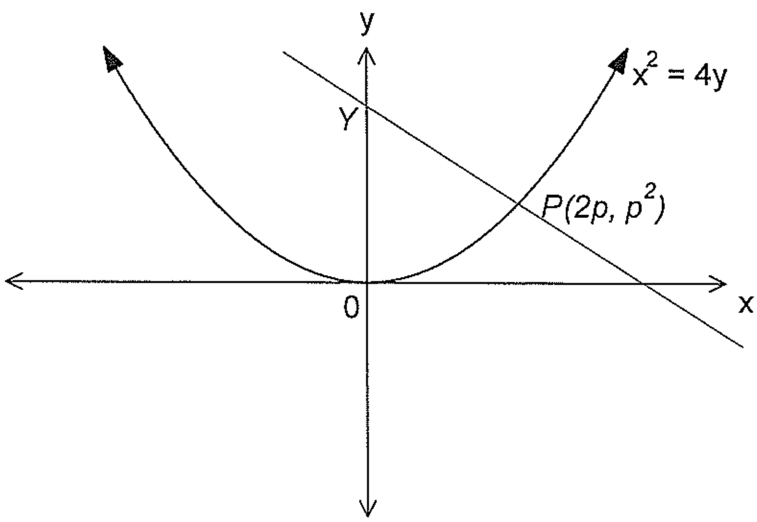
b) Show that the equation of the tangent at P is $y = px - ap^2$. **1**

c) The tangents at P and Q intersect at T. Find the coordinates of T. **2**

d) Given that the chord PQ has gradient 2, find the equation of the locus of T, and give a geometrical description of this locus. **2**

Question 6

$P(2p, p^2)$ is a variable point on the parabola $x^2 = 4y$



a) Show that the equation of the normal at P is $x + py = 2p + p^3$. **2**

b) The normal at P intersects the y-axis at Y . Write down the coordinates of Y. **1**

c) Find the value of p if if the area of ΔOPY is 33 square units, and p is a positive integer. **2**

Question 7

The polynomial $P(x) = Ax^3 + Bx^2 + 2Ax + C$ has real roots $\sqrt{p}, \frac{1}{\sqrt{p}}, \alpha$. Given that $C \neq 0$:

a) Explain why $\alpha = \frac{-C}{A}$ **1**

b) Show that $A^2 + C^2 = BC$ **2**

Question 8

a) Use the method of mathematical induction to prove that if x is a positive integer then $(1 + x)^n - 1$ is divisible by x for all positive integers n. **4**

b) Factorise $12^n - 4^n - 3^n + 1$. **1**

c) Hence deduce that $12^n - 4^n - 3^n + 1$ is divisible by 6 for all positive integers n. (Do NOT perform another proof by Mathematical Induction) **1**

End of task

YEAR 11 2010 EXT 1.

December Task Solns.

Q1.

a)

$$\begin{array}{r|l} & x^2 - 2 \\ x^2+1 & x^4 - x^2 + 1 \\ & x^4 + x^2 \\ \hline & -2x^2 + 1 \\ & -2x^2 - 2 \\ \hline & 3 \end{array}$$

1 correct quot. and remainder

$P(x) = (x^2+1)(x^2-2)+3$ 1

b) $P(x) = \frac{1}{4}(x+1)^3(x-2)^2$ (1, 1, 1)

Q2.

(i) $5/2$ 1

(ii) $\frac{\alpha\beta+\beta\gamma+\gamma\alpha}{\alpha\beta\gamma} = \frac{3/2}{-1/2}$ ← 1 ← 1

$= -3$

(iii) $(\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\beta\gamma+\alpha\gamma)$ 1

$= \left(\frac{5}{2}\right)^2 - 2\left(\frac{3}{2}\right)$

$= \frac{13}{4}$ 1

Q3.

a) i) $y = 3\left(\frac{x}{2}\right)^2$

$y = \frac{3x^2}{4}$ 1

ii) $\cos^2\theta + \sin^2\theta = 1$

$x^2 + (y+1)^2 = 1$ 1

b) $P(2) = 16 + 8 + 4p + 2q + 48 = 0$

$4p + 2q = -72$

$2p + q = -36$ ① 1

$P(-2) = 16 - 8 + 4p - 2q + 48 = 0$

$4p - 2q = -56$

$2p - q = -28$ ② 1

$4p = -64$

$p = -16$

$-32 + q = -36$

$q = -4$ 1

Q4.

If $n=1$ LHS $= 5^1 = 5$

RHS $= \frac{5}{4}(5-1) = \frac{5}{4} \times 4 = 5$

LHS = RHS ∴ True for $n=1$ 1

Assume true for $n=k$

ie. Assume

$$\sum_{r=1}^{k} 5^r = \frac{5}{4}(5^k - 1)$$

Need to prove true for $n=k+1$

ie. Prove

$$\sum_{r=1}^{k+1} 5^r = \frac{5}{4}(5^{k+1} - 1)$$

LHS $= 5^1 + 5^2 + 5^3 + \ldots + 5^k + 5^{k+1}$

$= \sum_{r=1}^{k} 5^r + 5^{k+1}$

$= \frac{5}{4}(5^k - 1) + 5^{k+1}$ 1

by assumption

$= \frac{1}{4}\cdot 5(5^k - 1) + 5^{k+1}$

$= \frac{1}{4}(5^{k+1}) - \frac{5}{4} + 5^{k+1}$

$= \frac{5}{4}(5^{k+1}) - \frac{5}{4}$ 1

$= \frac{5}{4}(5^{k+1} - 1)$

= RHS ∴ If true for $n=k$,

then also true for $n=k+1$.
Now statement is true for $n=1$
$\therefore$ True for $n=2,3,4\ldots$
By induction, it is true for all positive integers n. 1

Q5.

a) $m = \dfrac{ap^2-aq^2}{2ap-2aq} = \dfrac{a(p-q)(p+q)}{2a(p-q)}$

$= \dfrac{p+q}{2}$ 1

$y - ap^2 = \dfrac{(p+q)}{2}.(x-2ap)$ 1

$2y - 2ap^2 = (p+q)x - 2ap^2 - 2apq$

$2y = (p+q)x - 2apq$

$y = \dfrac{p+q}{2}.x - apq$

b) Let $q \to p$ to obtain the eqn of tangent: 1

$y \to \dfrac{2p}{2}.x - ap^2$

$y = px - ap^2$ is eqn of tangent

[OR $\dfrac{dy}{dx} = \dfrac{2x}{4a}$ $\quad \left\{ \begin{array}{l} x^2 = 4ay \\ y = x^2/4a \end{array} \right.$

$= \dfrac{x}{2a}$

$= \dfrac{2ap}{2a}$ at P

$= p$

$y - ap^2 = p(x-2ap)$

$y - ap^2 = px - 2ap^2$

$y = px - ap^2$]

c) $y = px - ap^2$
$y = qx - aq^2$

Intersect when

$px - ap^2 = qx - aq^2$

$px - qx = ap^2 - aq^2$

$(p-q)x = a(p-q)(p+q)$

$x = a(p+q)$ ---- 1

$y = p.a(p+q) - ap^2$

$= ap^2 + apq - ap^2$

$= apq$ ------- 1

$\therefore T\left(a(p+q), apq\right)$

d) Since $m=2$, $\dfrac{p+q}{2} = 2$

$p+q = 4$

$\therefore$ At T: $x = 4a$ which is constant.

$\therefore$ Locus is $x = 4a$ 1

But tangents can only intersect outside the parabola.
$\therefore$ Locus is the ray $x=4a$, lying below the parabola, (and parallel to the y-axis) 1 ← not needed.

["Ray" or "part of line" lying below or outside the parabola]

Q6.

a) $y = \frac{x^2}{4}$ $\quad \frac{dy}{dx} = \frac{2x}{4}$

$= \frac{2(2p)}{4}$ at P.

$= p$

Normal: $m = \frac{-1}{p}$ 1

$y - p^2 = -\frac{1}{p}(x - 2p)$ 1

$py - p^3 = -x + 2p$

$x + py = p^3 + 2p.$

b) If $x = 0$, $py = p^3 + 2p$

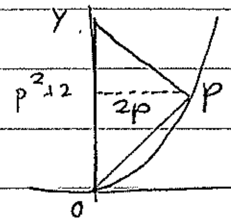
$y = p^2 + 2$

$\therefore Y(0, p^2 + 2)$ 1

c) Area

$= \frac{1}{2}\cdot(p^2+2).2p$

$= p^3 + 2p$ ---- 1



If $p^3 + 2p = 33$, $p^3 + 2p - 33 = 0$

($p = 1, 3, 11, 33$ – possible values)

$3^3 + 2(3) - 33 = 0$

$\therefore p = 3$ ---- 1

Q7.

a) Product of roots $= \sqrt{p}\cdot\frac{1}{\sqrt{p}}\cdot\alpha = \frac{-C}{A}$

$\therefore \alpha = \frac{-C}{A}.$ 1

b) α is a root

$A\left(-\frac{C}{A}\right)^3 + B\left(-\frac{C}{A}\right)^2 + 2A\left(-\frac{C}{A}\right) + C = 0$ 1

$\frac{-C^3}{A^2} + \frac{BC^2}{A^2} - 2C + C = 0$

$\frac{-C^3}{A^2} + \frac{BC^2}{A^2} = C$ 1

$-C^3 + BC^2 = A^2C$

$\div C \neq 0$

$-C^2 + BC = A^2$

ie. $A^2 + C^2 = BC.$

Q8.

a) If $n = 1$

$(1+x)^1 - 1 = 1 + x - 1 = x$

which is divisible by x

$\therefore$ True for $n = 1$ 1

Assume true for $n = k$: 1

$(1+x)^k - 1 = mx$ (m = integer)

Need to prove true for $n = k+1$

ie. Prove $(1+x)^{k+1} - 1 = px$

(p = integer)

LHS $= (1+x)(1+x)^k - 1$

$= (1+x)^k + x(1+x)^k - 1$

$= [(1+x)^k - 1] + x(1+x)^k$

$= mx + x(1+x)^k$

↖ by assumption 1

$= x(\underbrace{m + (1+x)^k}_{\text{integer}}) = xp.$ 1

$\therefore$ If true for $n = k$, then true for $n = k+1$

Conclusion .... (no mark)

b) $4^n(3^n - 1) - (3^n - 1)$

$= (4^n - 1)(3^n - 1)$ 1

c) $4^n - 1 = (1+3)^n - 1$ $\therefore$ Div by 3

$3^n - 1 = (1+2)^n - 1$ $\therefore$ Div by 2

$\therefore$ Product is div by $2 \times 3 = 6.$ 1